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STABILITY IN L^1 OF CIRCULAR VORTEX PATCHES

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Abstract. The motion of incompressible and ideal fluids is studied in the plane. The stability in L^1 of circular vortex patches is established among the class of all bounded vortex patches of equal strength.

For planar incompressible and ideal fluid flow, the theory of Yudovich [9] establishes global well-posedness of the initial value problem with initial vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Because vorticity is transported in 2d, it remains constant along particle trajectories. If Φ_t is the flow map, then the vorticity is given by $\omega(t, \Phi_t(y)) = \omega(0, y)$, for all $t > 0$ and $y \in \mathbb{R}^2$. When the initial vorticity is a patch of unit strength, represented by the indicator (characteristic) function I_{Ω_0} of a bounded open set $\Omega_0 \subset \mathbb{R}^2$, the resulting vorticity is I_{Ω_t}, with $\Omega_t = \Phi_t(\Omega_0)$. In the special case when Ω_0 is equal to a ball B, the patch is stationary, $\Phi_t(B) = B$, for all $t > 0$. Theorem 3, our main result, gives the stability in $L^1(\mathbb{R}^2)$ of any circular patch within the class of all bounded vortex patches of equal strength. No restriction is placed on the L^1 distance of the perturbation to the ball, and the flow region is not limited to a bounded domain, but rather is the entire space $\mathbb{R}^2$.

Wan and Pulvirenti [8] were the first to study stability of vortex patches in L^1. They considered the case where the flow was contained in a bounded region, although for the stability of circular patches they mention that this assumption can be removed. Their key estimate, **(J)**, shows that the total angular momenta of the patches can be used to control the L^1 difference between an arbitrary patch and a circular patch of the same total mass. They allow the strengths of the patches to differ, in which case the two patches are assumed to be close in L^1. Our generalization of their inequality, given in Lemma 2, estimates the L^1 distance of an arbitrary patch to a circular patch when both patches have equal strength. Stability in L^1, given in Theorem 3, follows immediately. Weaker stability results were given by Saffman [7] and Dritschel [4]. Dritschel controls the measure of the symmetric difference of two patches through a convenient integral, and this idea is incorporated into our argument in Lemma 1.

Stability in L^1 does not imply that the boundaries of the two patches remain close in any metric. Indeed, numerical simulations give strong evidence of fingering

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and filamentation; see [1, 3]. Spreading of vorticity may also occur. The best upper bound for the growth rate of the patch diameter is $\mathcal{O}(t\log t)^{1/4}$ given in [5]; see also [6]. Nevertheless, in spite of the fact that the patch geometry may be complicated, smoothness of smooth patch boundaries persists for all time; see [2].

For any bounded open set $A \subset \mathbb{R}^2$, denote its mass, momentum, and angular momentum by

$$|A| = \int_A dx, \quad M(A) = \int_A x\, dx, \quad \text{and} \quad i(A) = \int_A |x|^2 dx,$$

respectively. Our arguments depend heavily upon the fact that these three quantities are conserved in time when $A = \Omega_t$ is a patch moving with the flow.

Lemma 1. *If $A \subset \mathbb{R}^2$ is any bounded open set, then*

$$i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} \geq 0.$$

Equality holds if and only if the set A is a ball.

Proof. For any ball $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$, introduce the quantity

$$Q = Q(A; B_r(x_0)) = \int_{A\triangle B_r(x_0)} \left||x - x_0|^2 - r^2\right|\, dx, \tag{1}$$

in which $A\triangle B_r(x_0) = (A\setminus B_r(x_0)) \cup (B_r(x_0)\setminus A)$ denotes the symmetric difference. Note that $Q \geq 0$ and $Q = 0$ if and only if $A = B_r(x_0)$.

The quantity Q can also be written as

$$Q = \int_A (|x - x_0|^2 - r^2)\, dx + \int_{B_r(x_0)} (r^2 - |x - x_0|^2)\, dx,$$

since the portions of these two integrals over the set $A \cap B_r(x_0)$ cancel each other out.

Now, we can expand the first integral in Q and compute the second to obtain

$$Q = i(A) - 2x_0 \cdot M(A) + (|x_0|^2 - r^2)|A| + \frac{\pi}{2} r^4.$$

A rearrangement of terms gives

$$Q = i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} + \frac{1}{2\pi}\left(\pi r^2 - |A|\right)^2 + |A|\left|x_0 - \frac{M(A)}{|A|}\right|^2. \tag{2}$$

This last expression is minimized by choosing $B_r(x_0)$ with the same mass and center of mass as A:

$$|B_r(x_0)| = \pi r^2 = |A| \quad \text{and} \quad x_0 = \frac{M(A)}{|A|}.$$

With this choice, the lemma now follows. □

Lemma 2. *If $B = B_r(0)$, then for any bounded open set A,*

$$\|I_A - I_B\|_{L^1}^2 \leq 4\pi\, Q(A; B),$$

in which $Q(A; B)$ is defined by (1). Equality holds if and only if

$$A = B_a(0) \cup [B_b(0) \setminus B_r(0)], \tag{3}$$

with $a < r < b$ and $r^2 - a^2 = b^2 - r^2$.

Proof. Using the identity (2) and then Lemma 1, we have for any bounded open set A',

$$(|A'| - |B|)^2 = (|A'| - \pi r^2)^2 \leq 2\pi\, Q(A'; B), \tag{4}$$

with equality if and only if A' is a ball centered at the origin.

Next, we note that

$$\begin{aligned} \|I_A - I_B\|_{L^1}^2 &= |A \Delta B|^2 \\ &= (|A \setminus B| + |B \setminus A|)^2 \\ &\leq 2|A \setminus B|^2 + 2|B \setminus A|^2 \\ &= 2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2, \end{aligned}$$

with equality if and only if $|A \setminus B| = |B \setminus A|$.

Application of (4) with $A' = A \cup B$ and $A' = A \cap B$ yields

$$\begin{aligned} &2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2 \\ &\qquad \leq 4\pi\, [Q(A \cup B; B) + Q(A \cap B; B)] = 4\pi\, Q(A; B), \end{aligned}$$

with equality if and only if $A \cup B$ and $A \cap B$ are balls centered at the origin. This establishes the desired inequality.

The argument also shows that equality holds if and only if $A \cup B = B_b(0)$, $A \cap B = B_a(0)$, with $a < r < b$, and

$$|B_b(0) \setminus B| = |A \setminus B| = |B \setminus A| = |B \setminus B_a(0)|,$$

which gives (3). □

Theorem 3. *Let $B = B_r(0)$. Then for any bounded open set $\Omega_0 \subset \mathbb{R}^2$, we have that*

$$\|I_{\Omega_t} - I_B\|_{L^1}^2 \leq 4\pi \sup_{\Omega_0 \triangle B} \left||x|^2 - r^2\right| \, \|I_{\Omega_0} - I_B\|_{L^1},$$

for all $t > 0$.

Proof. The identity (2) shows that the quantity $Q(\Omega_t; B)$ depends only on conserved quantities, and it is therefore also conserved. In other words, we have $Q(\Omega_t; B) = Q(\Omega_0; B)$, for all $t > 0$. Thus, the result follows from Lemma 2 and the fact that

$$Q(\Omega_0; B) \leq \sup_{\Omega_0 \triangle B} \left||x|^2 - r^2\right| \, |\Omega_0 \triangle B| = \sup_{\Omega_0 \triangle B} \left||x|^2 - r^2\right| \, \|I_{\Omega_0} - I_B\|_{L^1}.$$

□

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